Functional van den Berg-Kesten-Reimer Inequalities and their Duals, with Applications

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Abstract

The BKR inequality conjectured by van den Berg and Kesten in [11], and proved by Reimer in [8], states that for A and B events on S, a finite product of finite sets $S_i, i = 1, \ldots, n$, and P any product measure on S,

$$P(A \Box B) \le P(A)P(B),$$

where the set $A \Box B$ consists of the elementary events which lie in both A and B for 'disjoint reasons.' Precisely, with $\mathbf{n} := \{1, \ldots, n\}$ and $K \subset \mathbf{n}$, for $\mathbf{x} \in S$ letting $[\mathbf{x}]_K = \{\mathbf{y} \in S : y_i = x_i, i \in K\}$, the set $A \Box B$ consists of all $\mathbf{x} \in S$ for which there exist disjoint subsets K and L of $\mathbf{n}$ for which $[\mathbf{x}]_K \subset A$ and $[\mathbf{x}]_L \subset B$.

The BKR inequality is extended to the following functional version on a general finite product measure space $(S, \mathbb{S})$ with product probability measure P,

$$E \left\{ \max_{\substack{K \cap L = \emptyset \\ K \subset \mathbf{n}, L \subset \mathbf{n}}} \underline{f}_K(\mathbf{X}) \underline{g}_L(\mathbf{X}) \right\} \le E\{f(\mathbf{X})\}\ E\{g(\mathbf{X})\},$$

where f and g are non-negative measurable functions, $\underline{f}_K(\mathbf{x}) = \operatorname{ess\,inf}_{\mathbf{y} \in [\mathbf{x}]_K} f(\mathbf{y})$ and $\underline{g}_L(\mathbf{x}) = \operatorname{ess\,inf}_{\mathbf{y} \in [\mathbf{x}]_L} g(\mathbf{y})$. The original BKR inequality is recovered by taking $f(\mathbf{x}) = \mathbf{1}_A(\mathbf{x})$ and $g(\mathbf{x}) = \mathbf{1}_B(\mathbf{x})$, and applying the fact that in general $\mathbf{1}_{A \Box B} \le \max_{K \cap L = \emptyset} \underline{f}_K(\mathbf{x}) \underline{g}_L(\mathbf{x})$.

Related formulations, and functional versions of the dual inequality on events by Kahn, Saks, and Smyth [6], are also considered. Applications include order statistics, assignment problems, and paths in random graphs.

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1 Introduction

This paper is a minor revision of [5], correcting an error in equation (5) that was pointed out to us by Richard Arratia during the preparation of [2], a draft of which he shared with us. Inequality (5) was earlier incorrectly stated as an equality. While the correction is important on its own, the error is inconsequential for the purposes of our original work. As (5) was only applied to show (20), for which we now provide a much simpler argument not involving (5), no results depended on its validity. In particular, the statements of all theorems here are the same as in [5].

For $\mathbf{x} = (x_1, \ldots, x_n) \in S$, where $S = \prod_{i=1}^n S_i$ any product space, and $K = \{k_1, \ldots, k_m\} \subseteq \mathbf{n} := \{1, \ldots, n\}$ with $k_1 < \cdots < k_m$, define

$$\mathbf{x}_K = (x_{k_1}, \ldots, x_{k_m}) \quad \text{and} \quad [\mathbf{x}]_K = \{\mathbf{y} \in S : y_K = x_K\},$$

the restriction of $\mathbf{x}$ to the indicated coordinates, and the collection of all elements in S which agree with $\mathbf{x}$ in those coordinates, respectively. For $A, B \subseteq S$ we say that $\mathbf{x} \in A, \mathbf{y} \in B$ *disjointly* if there exist

$$K, L \subseteq \mathbf{n}, K \cap L = \emptyset \text{ such that} \quad [\mathbf{x}]_K \subseteq A \quad \text{and} \quad [\mathbf{y}]_L \subseteq B, \tag{1}$$

and denote

$$A \Box B = \{\mathbf{x} : \mathbf{x} \in A, \mathbf{x} \in B \text{ disjointly}\}. \tag{2}$$

The operation $A \Box B$ corresponds to elementary events which are in both A and B for disjoint 'reasons' in the sense that inclusion in A and B is determined on disjoint sets of coordinates.

Theorem 1.1 was conjectured in van den Berg and Kesten [11]. It was proved in [11] for A and B increasing sets and $S = \{0,1\}^n$, and it was also demonstrated there that Theorem 1.1 follows from its special case $S = \{0,1\}^n$. Using the latter fact, the conjecture was established in general by Reimer [8].

Theorem 1.1. *For $P = \prod_{i=1}^n P_i$ any product measure on $S = \prod_{i=1}^n S_i$, S_i finite,*

$$P(A \Box B) \leq P(A)P(B). \tag{3}$$

Many useful formulations can be found in van den Berg and Fiebig [10], in addition to the following motivating example which appeared earlier in [11]. Independently assign a random direction to each edge $e = \{v_i, v_j\}$ of a finite graph, with $p_e(v_i, v_j) = 1 - p_e(v_j, v_i)$ the probability of the edge e being directed from vertex v_i to v_j. With V_1, V_2, W_1, W_2 sets of vertices, Theorem 1.1 yields that the product of the probabilities that there exist directed paths from V_1 to V_2 (event A) and from W_1 to W_2 (event B) is an upper bound to the probability that there exist two disjoint directed paths, one from V_1 to V_2 and another from W_1 to W_2 (event $A \Box B$).

The main thrust of this paper is to show how Theorem 1.1 implies inequalities in terms of functions, of which (3) is the special case of indicators, and similarly for the dual inequality of [6]. These functional inequalities, and their duals, are stated in Theorems 1.2 and 1.5, and their proofs can be found in Section 3. Applications to order statistics, allocation problems, and random graphs are given in Section 2. Specializing to monotone functions, we derive

related inequalities and stochastic orderings in Section 4; these latter results are connected to those of Alexander [1].

For each $i = 1, \ldots, n$, let $(S_i, \mathbb{S}_i)$ be measurable spaces, and set $S = \prod_{i=1}^n S_i$ and $\mathbb{S} = \bigotimes_{i=1}^n \mathbb{S}_i$, the product sigma algebra. Henceforth, all given real valued functions on S, such as $f_\alpha, g_\beta, \alpha \in \mathcal{A}, \beta \in \mathcal{B}$ are assumed to be $(\mathbb{S}, \mathbb{B})$ measurable where $\mathbb{B}$ denotes the Borel sigma algebra of $\mathbf{R}$, and functions on S with values in $2^{\mathbf{n}}$, such as $K(\mathbf{x})$ in inequality (d) of Theorem 1.2 below, are assumed to be $(\mathbb{S}, 2^{2^{\mathbf{n}}})$ measurable. Measurability issues arise in definitions (9), (4), and (16), and are settled in Section 5. We also show in Section 5 that Theorem 1.2 applies to the completion of the measure space $(S, \mathbb{S})$ with respect to the measure P appearing in the theorem; similarly for Theorem 1.5.

For $K \subseteq \mathbf{n}$ we say that a function f defined on S depends on K if $\mathbf{x}_K = \mathbf{y}_K$ implies $f(\mathbf{x}) = f(\mathbf{y})$. The inequalities in Theorems 1.2 and 1.5 require one of two frameworks, the first of which is the following.

Framework 1. *$\{f_\alpha(\mathbf{x})\}_{\alpha \in \mathcal{A}}$ and $\{g_\beta(\mathbf{y})\}_{\beta \in \mathcal{B}}$ are given collections of non-negative functions on S, such that f_α, g_β depend respectively on subsets of $\mathbf{n}$ K_α, L_β in $\mathcal{K} = \{K_\alpha\}_{\alpha \in \mathcal{A}}$ and $\mathcal{L} = \{L_\beta\}_{\beta \in \mathcal{B}}$, where $\mathcal{A}$ and $\mathcal{B}$ are finite or countable.*

The elements of $\mathcal{K}$ and $\mathcal{L}$ are not assumed to be distinct; we may have, say, $K_\alpha = K_\gamma$ for some $\alpha \neq \gamma$ and $f_\alpha \neq f_\gamma$. Note also that if a function depends on K, it depends on any subset of $\mathbf{n}$ containing K. For notational brevity we may write α for K_α; for example, we may use $\alpha \cap \beta$ as an abbreviation for $K_\alpha \cap L_\beta$, and also $\mathbf{x}_\alpha$ for $\mathbf{x}_{K_\alpha}$.

The second framework is

Framework 2. *f and g are two given non-negative functions, and $\mathcal{K}$ and $\mathcal{L}$ are any subsets of $2^{\mathbf{n}}$. With P a probability measure on $(S, \mathbb{S})$ define for $K \in \mathcal{K}, L \in \mathcal{L}$,*

$$\underline{f}_K(\mathbf{x}) = \operatorname{ess} \inf_{\mathbf{y} \in [\mathbf{x}]_K} f(\mathbf{y}), \quad \textit{and} \quad \underline{g}_L(\mathbf{x}) = \operatorname{ess} \inf_{\mathbf{y} \in [\mathbf{x}]_L} g(\mathbf{y}), \tag{4}$$

where the essential infimums for $\underline{f}_K(\mathbf{x})$ and $\underline{g}_L(\mathbf{x})$ are being taken with respect to the product probability measure on the coordinates in K^c and L^c respectively.

Our functional extension of the BKR inequality (3) is

Theorem 1.2. *Let $\mathbf{X} = (X_1, \ldots, X_n) \in S$ be a random vector and P a probability measure on $(S, \mathbb{S})$ under which $X_1, \ldots, X_n$ are independent.*

1. *Under framework 1,*

$$E\left\{ \sup_{\alpha \cap \beta = \emptyset} f_\alpha(\mathbf{X}) g_\beta(\mathbf{X}) \right\} \leq E\left\{ \sup_\alpha f_\alpha(\mathbf{X}) \right\} E\left\{ \sup_\beta g_\beta(\mathbf{X}) \right\}. \tag{a}$$

2. *Under framework 2,*

$$E\left\{ \max_{\substack{K \cap L = \emptyset \\ K \in \mathcal{K}, L \in \mathcal{L}}} \underline{f}_K(\mathbf{X}) \underline{g}_L(\mathbf{X}) \right\} \leq E\left\{ f(\mathbf{X}) \right\} E\left\{ g(\mathbf{X}) \right\}. \tag{b}$$

The special case of (b) where $\mathcal{K} = \mathcal{L} = 2^{\mathbf{n}}$, the collection of all subsets of $\mathbf{n}$, clearly implies the inequality in general.

In [8], inequality (3) for the $\Box$ operation was proven only for discrete finite product spaces, that is, a finite product of finite sets; Theorem 1.2 applies to functions of a vector $\mathbf{X}$ having independent coordinates taking values in *any* measure space. For $f(\mathbf{x}) = \mathbf{1}_A(\mathbf{x})$ and $g(\mathbf{x}) = \mathbf{1}_B(\mathbf{x})$ for $A, B \in \mathbb{S}$, we have

$$\mathbf{1}_{A\Box B} \leq \max_{K\cap L=\emptyset} \underline{f}_K(\mathbf{x})\underline{g}_L(\mathbf{x}). \tag{5}$$

To see (5), note that replacing essential infimum by infimum in (4), the inequality becomes equality. Hence (5) holds as stated because the essential infimum is at least as large as the infimum. In other words, elements of $A\Box B$ demand disjoint 'reasons' for A and B that hold for all outcomes in the probability space, while the right hand side of (5) only requires that the 'reasons' be almost sure.

In [5], the specialization of (b) to indicator functions and $\mathcal{K} = \mathcal{L} = 2^{\mathbf{n}}$, and 'equality' in (5), was interpreted to mean that inequality (3) holds for general product spaces. However, as (b) is an inequality, this interpretation now yields that for all $A, B \in \mathbb{S}$ the set in $\mathbb{S}$ whose indicator appears in the right hand side of (5) contains $A\Box B$ and has probability bounded above by the product of the probabilities of A and B. This latter interpretation appears in Corollary 4 of [2] when $\mathbb{S}$ is taken to be Euclidean space.

The following is a straightforward generalization of Theorem 1.2, stated here only for inequality (a). Note that in the inequality below, as m increases the pairwise constraints $\alpha_i \cap \alpha_j = \emptyset$ become more restrictive, and the inequality less sharp.

Theorem 1.3. *Let $\mathbf{X} \in S$ be a random vector with independent coordinates. Then for given finite or countable collections of non-negative functions $\{f_{i,\alpha}\}_{\alpha\in\mathcal{A}_i}$ depending on $\{K_{i,\alpha}\}_{\alpha\in\mathcal{A}_i}$, $i = 1, \ldots, m$,*

$$E\left\{\sup_{\substack{(\alpha_1,\ldots,\alpha_m)\in\mathcal{A}_1\times\cdots\times\mathcal{A}_m \\ \alpha_k\cap\alpha_l=\emptyset,\ k\neq l}} \prod_{i=1}^m f_{i,\alpha_i}(\mathbf{X})\right\} \leq \prod_{i=1}^m E\left\{\sup_{\alpha\in\mathcal{A}_i} f_{i,\alpha}(\mathbf{X})\right\}.$$

Next we describe an inequality of Kahn, Saks, and Smyth [6], which may be considered dual to the BKR inequality (3), and then provide a function version. We use a notation compatible with (3). With 'disjointly' defined in (1), denote

$$A\Diamond B = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in A, \mathbf{y} \in B \,\text{disjointly}\}.$$

Note that

$$\mathbf{1}_{A\Box B}(\mathbf{x}) = \mathbf{1}_{A\Diamond B}(\mathbf{x}, \mathbf{x}).$$

The following, which we call the KSS inequality, is dual to Theorem 1.1 and is given in [6].

Theorem 1.4. *If P denotes the uniform measure over $\{0, 1\}^n$, then for any $(A, B) \subseteq \{0, 1\}^n \times \{0, 1\}^n$,*

$$(P \times P)(A\Diamond B) \leq P(A \cap B). \tag{6}$$

Our functional extension of the KSS inequality is as follows.

Theorem 1.5. *Let $\mathbf{X} = (X_1, \ldots, X_n) \in S$ be a random vector, P any probability measure on $(S, \mathbb{S})$ such that $X_1, \ldots, X_n$ are independent, and $\mathbf{Y}$ an independent copy of $\mathbf{X}$.*

1. *Under framework 1,*

$$E\left\{\sup_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{Y})\right\} \le E\left\{\sup_{\alpha,\beta} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\}. \qquad (a')$$

2. *Under framework 2,*

$$E\left\{\max_{\substack{K\cap L=\emptyset\\ K\in\mathcal{K}, L\in\mathcal{L}}} \underline{f}_K(\mathbf{X})\underline{g}_L(\mathbf{Y})\right\} \le E\left\{f(\mathbf{X})g(\mathbf{X})\right\}. \qquad (b')$$

The $\Diamond$ operation in Theorem 1.4 on $\{0,1\}^n \times \{0,1\}^n$ was defined implicitly in [6], and the inequality was extended there to product measure on discrete finite product spaces. With $f(\mathbf{x})$ and $g(\mathbf{x})$ the indicator functions of sets A and B respectively, we have the inequality

$$\mathbf{1}_{A\Diamond B}(\mathbf{x},\mathbf{y}) \le \max_{K\cap L=\emptyset} \underline{f}_K(\mathbf{x})\underline{g}_L(\mathbf{y}), \qquad (7)$$

where $\underline{f}_K, \underline{g}_L$ are given in (4). Therefore, inequality (b$'$) of Theorem 1.5 specialized to the case where $\mathcal{K} = \mathcal{L} = 2^{\mathbf{n}}$ and f and g are indicators says that the original KSS inequality (6) for events in discrete finite product spaces extends to vectors having independent coordinates taking values in any measure space in the sense that $A\Diamond B$ is a subset of the set whose indicator is the function appearing on the right hand side of (7), and has probability bounded by $P(A\cap B)$, by (b').

We next discuss further formulations of Theorems 1.2 and 1.5 which are of independent interest, and will be used in the proof. Under Framework 1, for any subsets K and L of $\mathbf{n}$, define

$$\tilde{f}_K(\mathbf{x}) = \sup_{\alpha: K_\alpha\subseteq K} f_\alpha(\mathbf{x}) \quad \text{and} \quad \tilde{g}_L(\mathbf{x}) = \sup_{\beta: L_\beta\subseteq L} g_\beta(\mathbf{x}). \qquad (8)$$

For any given functions $K(\mathbf{x})$ and $L(\mathbf{x})$ defined on S and taking values in $2^{\mathbf{n}}$, under Framework 1, extend (8) to

$$\tilde{f}_{K(\mathbf{x})}(\mathbf{x}) = \sup_{\alpha: K_\alpha\subseteq K(\mathbf{x})} f_\alpha(\mathbf{x}) \quad \text{and} \quad \tilde{g}_{L(\mathbf{x})}(\mathbf{x}) = \sup_{\beta: L_\beta\subseteq L(\mathbf{x})} g_\beta(\mathbf{x}), \qquad (9)$$

and under Framework 2, extend (4) to

$$\underline{f}_{K(\mathbf{x})}(\mathbf{x}) = \operatorname{ess}\inf_{\mathbf{y}\in[\mathbf{x}]_{K(\mathbf{x})}} f(\mathbf{y}), \quad \text{and} \quad \underline{g}_{L(\mathbf{x})}(\mathbf{x}) = \operatorname{ess}\inf_{\mathbf{y}\in[\mathbf{x}]_{L(\mathbf{x})}} g(\mathbf{y}). \qquad (10)$$

Proposition 3.1 shows that parts (c) and (d) of Proposition 1.1 are reformulations of (a) of Theorem 1.2, and likewise (e) a reformulation of (b).

Proposition 1.1. *Let the hypotheses of Theorem 1.2 hold. In Framework 1 (c) and (d) below obtain.*

$$E\left\{\max_{\substack{K\cap L=\emptyset\\ K\in\mathcal{K},L\in\mathcal{L}}} \tilde{f}_K(\mathbf{X})\tilde{g}_L(\mathbf{X})\right\} \le E\left\{\sup_{\alpha} f_\alpha(\mathbf{X})\right\} E\left\{\sup_{\beta} g_\beta(\mathbf{X})\right\}. \tag{c}$$

$$E\left\{\tilde{f}_{K(\mathbf{X})}(\mathbf{X})\tilde{g}_{L(\mathbf{X})}(\mathbf{X})\right\} \le E\left\{\sup_{\alpha} f_\alpha(\mathbf{X})\right\} E\left\{\sup_{\beta} g_\beta(\mathbf{X})\right\}, \tag{d}$$

holding for any given $K(\mathbf{x}) \in \mathcal{K}$ *and* $L(\mathbf{x}) \in \mathcal{L}$ *such that* $K(\mathbf{x}) \cap L(\mathbf{x}) = \emptyset$.

In Framework 2 we have

$$E\left\{\underline{f}_{K(\mathbf{X})}(\mathbf{X})\underline{g}_{L(\mathbf{X})}(\mathbf{X})\right\} \le E\left\{f(\mathbf{X})\right\} E\{g(\mathbf{X})\}, \tag{e}$$

holding for any given $K(\mathbf{x}) \in \mathcal{K}$ *and* $L(\mathbf{x}) \in \mathcal{L}$ *such that* $K(\mathbf{x}) \cap L(\mathbf{x}) = \emptyset$.

As in Theorem 1.2 the special cases of (c) and (d) where $\mathcal{K}$ and $\mathcal{L}$ both equal $2^{\mathbf{n}}$ implies the inequality in general. Similarly, the special case of inequality (e) with $\mathcal{K} = \mathcal{L} = 2^{\mathbf{n}}$ and $L(\mathbf{x}) = K^c(\mathbf{x})$, where K^c denotes the complement of K, yields the inequality in general, that is, (e) is equivalent to the statement that for any given $K(\mathbf{x})$,

$$E\left\{\underline{f}_{K(\mathbf{X})}(\mathbf{X})\underline{g}_{K^c(\mathbf{X})}(\mathbf{X})\right\} \le E\left\{f(\mathbf{X})\right\} E\{g(\mathbf{X})\}.$$

Parallel to the claims of Proposition 1.1, parts (c$'$) and (d$'$) below are reformulations of (a$'$) of Theorem 1.5, and (e$'$) a reformulation of (b$'$) . For given f and a function $K(\mathbf{x},\mathbf{y})$ taking values in $2^{\mathbf{n}}$, define $\tilde{f}_{K(\mathbf{X},\mathbf{Y})}$ and $\underline{f}_{K(\mathbf{X},\mathbf{Y})}$ by replacing $K(\mathbf{x})$ by $K(\mathbf{x},\mathbf{y})$ in (9) and (10) respectively.

Proposition 1.2. *Let the hypotheses of Theorem 1.5 hold. In Framework 1 (c$'$) and (d$'$) below obtain.*

$$E\left\{\max_{K\in\mathcal{K},L\in\mathcal{L}} \tilde{f}_K(\mathbf{X})\tilde{g}_L(\mathbf{Y})\right\} \le E\left\{\sup_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\}. \tag{c$'$}$$

$$E\left\{\tilde{f}_{K(\mathbf{X},\mathbf{Y})}(\mathbf{X})\tilde{g}_{L(\mathbf{X},\mathbf{Y})}(\mathbf{Y})\right\} \le E\left\{\sup_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\}, \tag{d$'$}$$

holding for any given $K(\mathbf{x},\mathbf{y}) \in \mathcal{K}$ *and* $L(\mathbf{x},\mathbf{y}) \in \mathcal{L}$ *replacing* $K(\mathbf{x})$ *and* $L(\mathbf{x})$ *in (9), respectively, and satisfying* $K(\mathbf{x},\mathbf{y}) \cap L(\mathbf{x},\mathbf{y}) = \emptyset$.

In Framework 2 we have

$$E\left\{\underline{f}_{K(\mathbf{X},\mathbf{Y})}(\mathbf{X})\underline{g}_{L(\mathbf{X},\mathbf{Y})}(\mathbf{Y})\right\} \le E\left\{f(\mathbf{X})g(\mathbf{X})\right\}, \tag{e$'$}$$

for given $K(\mathbf{x},\mathbf{y}) \in \mathcal{K}$ *and* $L(\mathbf{x},\mathbf{y}) \in \mathcal{L}$ *replacing* $K(\mathbf{x})$ *and* $L(\mathbf{x})$ *in (10), respectively.*

2 Applications

Example 2.1. ***Order Statistics Type Inequalities*** *Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of independent non-negative random variables with associated order statistics $X_{[n]} \leq \cdots \leq X_{[1]}$. Let $\mathcal{A} = \mathcal{B}$ be the collection of all the singletons $\alpha \in \mathbf{n}$ and $f_\alpha(\mathbf{x}) = g_\alpha(\mathbf{x}) = x_\alpha$. Then*

$$\max_\alpha f_\alpha(\mathbf{X}) = X_{[1]}, \quad \max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X}) g_\beta(\mathbf{X}) = X_{[1]}X_{[2]},$$

and inequality (a) of Theorem 1.2 provides the middle inequality in the string

$$EX_{[1]}EX_{[2]} \leq EX_{[1]}X_{[2]} \leq (EX_{[1]})^2 \leq EX_{[1]}^2.$$

The leftmost inequality is true since order statistics are always positively correlated (moreover they are associated as defined by Esary et al [3], and even MTP_2, see Karlin and Rinott [7]); the rightmost inequality follows from Jensen.

Theorem 1.2 allows a large variety of extensions of this basic order statistics inequality. For example, taking $\mathcal{A}$ and $\mathcal{B}$ to be all k and l subsets of $\mathbf{n}$ respectively, then with

$$f_\alpha(\mathbf{x}) = \prod_{j\in\alpha} x_j \quad \textit{and} \quad g_\beta(\mathbf{x}) = \prod_{j\in\beta} x_j \tag{11}$$

we derive

$$E\left(\prod_{j=1}^{k+l} X_{[j]}\right) \leq E\left(\prod_{j=1}^{k} X_{[j]}\right) E\left(\prod_{j=1}^{l} X_{[j]}\right).$$

Dropping the non-negativity assumption on $X_1, \ldots, X_n$, we have for all $t > 0$,

$$Ee^{t(X_{[1]}+X_{[2]})} \leq [Ee^{tX_{[1]}}]^2 = Ee^{tX_{[1]}}Ee^{tY_{[1]}} = Ee^{t(X_{[1]}+Y_{[1]})},$$

with Y_i's being independent copies of the X_i's. Likewise, for all $t > 0$,

$$Ee^{-t(X_{[n]}+X_{[n-1]})} \leq [Ee^{-tX_{[n]}}]^2 = Ee^{-tX_{[n]}}Ee^{-tY_{[n]}} = Ee^{-t(X_{[n]}+Y_{[n]})}.$$

Moment generating function and Laplace orders are discussed in Shaked and Shanthikumar [9].

Returning to non-negative variables, a variation of (11) follows by replacing products with sums, that is,

$$f_\alpha(\mathbf{x}) = \sum_{j\in\alpha} x_j \quad \textit{and} \quad g_\beta(\mathbf{x}) = \sum_{j\in\beta} x_j, \tag{12}$$

which for, $k = l = 2$ say, yields

$$E \max_{\{i,j,k,l\}=\{1,2,3,4\}} (X_{[i]} + X_{[j]})(X_{[k]} + X_{[l]}) \leq [E(X_{[1]} + X_{[2]})]^2.$$

Though the maximizing indices on the left hand side will be $\{1, 2, 3, 4\}$ as indicated, the choice is not fixed and depends on the X's; note, for example, that $(X_{[1]} + X_{[2]})(X_{[3]} + X_{[4]})$ is never maximal apart from degenerate cases.

Definition (11) and (12) are special cases where f and g are increasing non-negative functions of k and l variables and

$$f_\alpha(\mathbf{x}) = f(\mathbf{x}_\alpha) \quad \textit{and} \quad g_\beta(\mathbf{x}) = g(\mathbf{x}_\beta); \tag{13}$$

when f and g are symmetric,

$$E \max_{\{i_1,\ldots,i_k,j_1,\ldots,j_l\}=\{1,\ldots,k+l\}} f(X_{[i_1]},\ldots,X_{[i_k]})g(X_{[j_1]},\ldots,X_{[j_l]}) \leq Ef(X_{[1]},\ldots,X_{[k]})Eg(X_{[1]},\ldots,X_{[l]}).$$

We now give an example which demonstrates that these order statistics type inequalities can be considered in higher dimensions. Let $\mathbf{X}_1,\ldots,\mathbf{X}_n$ be independent vectors in $\mathbf{R}^m$, and for $\alpha,\beta \subseteq \mathbf{n}$ with $|\alpha| = |\beta| = 3$ let f_α and g_β be given as in (13), where $f(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = g(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ is, say, the area of the triangle formed by the given three vectors. Theorem 1.2 gives that the expected greatest product of the areas of two triangles with distinct vertices is bounded above by the square of the expectation of the largest triangular area.

To explore the dual inequality in these settings, let $\mathbf{X}$ be a vector of independent variables with support contained in $[0,1]$, and $\mathbf{Y}$ an independent copy. With $\mathcal{A} = \mathcal{B}$ the collections of all singletons α in $\mathbf{n}$, and $f_\alpha(\mathbf{x}) = x_\alpha, g_\beta(\mathbf{x}) = 1 - x_\beta$, inequality (a′) of Theorem 1.5 gives

$$E\left\{\max_{\alpha\neq\beta} X_\alpha(1-Y_\beta)\right\} \leq EX_{[1]}(1-X_{[n]}). \tag{14}$$

Note that $\max_{\alpha\neq\beta} X_\alpha(1-Y_\beta) \neq X_{[1]}(1-Y_{[n]})$; the right hand side might be larger because of the restriction $\alpha \neq \beta$. Removing the restriction $\alpha \neq \beta$ reverses (14), that is,

$$EX_{[1]}(1-X_{[n]}) \leq EX_{[1]}E(1-X_{[n]}) = EX_{[1]}E(1-Y_{[n]}) = EX_{[1]}(1-Y_{[n]}) = E\left\{\max_{\alpha,\beta} X_\alpha(1-Y_\beta)\right\},$$

where the inequality follows by the negative association of $X_{[1]}$ and $1 - X_{[n]}$.

Following our treatment of applications of Theorem 1.2 we can extend (14) as follows: with $\mathcal{A}$ and $\mathcal{B}$ the collection of all k and l subsets of $\mathbf{n}$ respectively, and

$$f_\alpha(\mathbf{x}) = \prod_{j\in\alpha} x_j \quad \textit{and} \quad g_\beta(\mathbf{x}) = \prod_{j\in\beta}(1-x_j),$$

we obtain

$$E\left\{\max_{\alpha\cap\beta=\emptyset} \prod_{i\in\alpha,j\in\beta} X_i(1-Y_j)\right\} \leq E\left\{\prod_{1\leq i\leq k,1\leq j\leq l} X_{[i]}(1-X_{[n-j+1]})\right\}.$$

We now consider resource allocation problems of the following type. Suppose that two projects A and B have to be completed using n available resources represented by the components of a vector $\mathbf{x}$. Each resource can be used for at most one project, and an allocation is given by a specification of disjoint subsets of resources. For any given subsets $\alpha,\beta \subseteq \mathbf{n}$, let $f_\alpha(\mathbf{x})$ and $g_\beta(\mathbf{x})$ count the number of ways that projects A and B can be completed using the resources $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ respectively. The exact definitions of the projects and the counts are immaterial; in particular larger sets do not necessarily imply more ways to

carry out a project. For an allocation α, β, $\alpha \cap \beta = \emptyset$, the total number of ways to carry out the two projects together is the product $f_\alpha(\mathbf{x})g_\beta(\mathbf{x})$. When the resources are independent variables, inequality (a) of Theorem 1.2 bounds the expected maximal number of ways of completing A and B together, by the product of the expectations of the maximal number of ways of completing each project alone. The bound is simple in the sense that it does not require understanding of the relation between the two projects. In particular, it can be computed without knowledge of the optimal allocation of resources.

Example 2.2. *With J a list of tasks, consider fulfilling the set of tasks on (not necessarily disjoint) lists $A \subseteq J$ and $B \subseteq J$, in two distant cities using disjoint sets of workers chosen from $1, 2, \ldots, n$. Each worker may be sent to one of the cities and assigned a single task or a set of tasks which he can perform. A worker may be qualified to fill more than one set of tasks. For $i \in \mathbf{n}$, let $\mathbf{x}_i \subseteq 2^J$ be the collection of possible assignments of tasks for worker i, (that is, the sets of tasks worker i is qualified to fulfill.); For $\alpha, \beta \subseteq \mathbf{n}$ and $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, let $f_\alpha(\mathbf{x})$ equal the number of ways the collection of workers α can complete A, and $g_\beta(\mathbf{x})$ the number of ways the collection β can complete B. When the qualifications $\mathbf{X}_i, i \in \mathbf{n}$ are independent, Theorem 1.2 bounds the expectation of the maximal number of ways of fulfilling the task requirements in both cities, by the product of the expectations of the maximal numbers of ways that the requirements in each collection can be separately satisfied.*

Example 2.3. ***Paths on Graphs*** *Consider a graph $\mathcal{G}$ with an arbitrary fixed vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$, where for each pair of vertices the existence of the edge $\{v_i, v_j\}$ is determined independently using a probability rule based on v_i, v_j, perhaps depending only on $d(\{v_i, v_j\})$ for some function d. Let $\mathbf{X} = \{X_{\{i,j\}}\}$ where $X_{\{i,j\}}$ is the indicator that there exists an edge between v_i and v_j. For instance, with $\mathcal{V} \subseteq \mathbf{R}^m$ and $Z_{\{i,j\}}, 1 \leq i, j \leq n$ independent non-negative variables, we may take for $v_i, v_j \in \mathcal{V}$,*

$$X_{\{i,j\}} = \mathbf{1}(d(\{v_i, v_j\}) < Z_{\{i,j\}}) \quad \text{where} \quad d(\{v_i, v_j\}) = ||v_i - v_j|| \quad \text{(Euclidean distance)} \quad .$$

Note that since the variables $Z_{\{i,j\}}$ do not have to be identically distributed, we can set $Z_{i,i} = 0$ and avoid self loops should we wish to do so.

Let a path in the graph $\mathcal{G}$ from u to w be any ordered tuple of vertices $v_{i_1}, \ldots, v_{i_p}$ with $v_{i_1} = u, v_{i_p} = w$ and $X_{\{i_k, i_{k+1}\}} = 1$ for $k = 1, \ldots, p-1$, and having all edges $\{v_{i_k}, v_{i_{k+1}}\}$ distinct. For u, v and w in $\mathcal{V}$ and $\alpha, \beta \subseteq \{\{i,j\} : 1 \leq i, j \leq n\}$, let $f_\alpha(\mathbf{X})$ be the number of paths in the graph from u to v which use only edges $\{v_i, v_j\}$ for $\{i,j\} \in \alpha$; in the same manner, let $g_\beta(\mathbf{X})$ be the number of paths in the graph from v to w which use only edges $\{v_i, v_j\}$ for $\{i,j\} \in \beta$.

The "projects" A and B in this framework are to create paths from u to v using α, and from v to w using β, respectively, which combine together, when $\alpha \cap \beta = \emptyset$, to give the overall project of creating a path from u to w passing through v. As the product $f_\alpha(\mathbf{X})g_\beta(\mathbf{X})$ for $\alpha \cap \beta = \emptyset$ is the number of paths from u to w via v for the given allocation, Theorem 1.2 provides a bound on the expected maximal number of such paths over all allocations in terms of the product of the expectations of the maximal number of paths from u to w and from w to v when the paths are created separately. Though finding the optimal allocation may be demanding, the upper bound can be computed simply, for this case in particular by monotonicity of $f_\alpha(\mathbf{x}), g_\beta(\mathbf{x})$ in α and β for fixed $\mathbf{x}$, implying that the maximal number of paths created separately is attained when using all available edges, i.e. at $\alpha = \beta = \mathbf{n}$.

However, the result and the upper bound hold even in constrained situations where the existence of more edges does not lead to more paths, that is, in cases where the functions f_α, g_β are not monotone in α and β. One such case would be where the existence of a particular edge mandates that all paths from u to v use it. More specifically, for some fixed $\{i_0, j_0\}$ suppose that if $\{i_0, j_0\} \in \alpha$ and $x_{\{i_0,j_0\}} = 0$ then $f_\alpha(\mathbf{x})$ counts the number of paths from u to v. On the other hand if $x_{\{i_0,j_0\}} = 1$ then $f_\alpha(\mathbf{x})$ counts the number of paths from u to v which use the edge $\{v_{i_0}, v_{j_0}\}$. In general such f_α will not be monotone.

This example easily generalizes to paths with multiple waypoints. We may also consider directed graphs where for $1 \le i \ne j \le n$ the directed edge (v_i, v_j) from v_i to v_j exists when $X_{ij} = 1$, the directed edge (v_j, v_i) from v_j to v_i exists when $X_{ij} = -1$ and $X_{ij} = 0$ when no edge exists. Returning to the graph example following the statement of Theorem 1.1, when the signed edge indicators $\{X_{ij}\}_{1\le i<j\le n}$ are independent, inequality (a) of Theorem 1.2 provides a bound on the expected maximal number of paths from vertices v_1 to v_2 and w_1 to w_2 using disjoint edges. Another possible extension is to consider paths between subsets of vertices.

For application of the dual inequality, consider for example two directed graphs on the same vertex set, determined by equally distributed and independent collections of signed edge indicators $\mathbf{X}$ and $\mathbf{Y}$, each having independent (but not necessarily identically distributed) components. Let $\alpha, \beta \subseteq \{(i,j) : 1 \le i \ne j \le n\}$, and $f_\alpha(\mathbf{X})$ be the number of directed paths in the graph from vertices u to v which use only $\mathbf{X}$ edges (v_i, v_j) with $(i,j) \in \alpha$; in the same manner, let $g_\beta(\mathbf{Y})$ be the number of directed paths in the graph from v back to u which use only $\mathbf{Y}$ edges (v_i, v_j) with $(i,j) \in \beta$. Consider the expected maximal number of paths, over all α and β with $\alpha \cap \beta = \emptyset$, that go from u to v using the $\mathbf{X}$ edges α and return to u from v using the $\mathbf{Y}$ edges β. Then Theorem 1.5 implies that this expectation is bounded by the expected maximal number of paths, over all α and β, to move from u to v using α, and then returning to u using β, all with $\mathbf{X}$ edges, but where edges used on the forward trip may now also be used for the return.

3 Proofs

3.1 Proofs of Proposition 1.1 and Theorem 1.2

We first reduce the problem by proving the following implications between the parts of Theorems 1.2 and Proposition 1.1.

Proposition 3.1. *(a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and (b) $\Leftrightarrow$ (e).*

Proof: *(a) $\Rightarrow$ (c)*: Apply inequality (a) to the finite collections $\{\tilde{f}_K\}_{K\in\mathcal{K}}, \{\tilde{g}_L\}_{L\in\mathcal{L}}$ and use

$$\sup_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{x})g_\beta(\mathbf{x}) = \max_{\substack{K\cap L=\emptyset\\ K\in\mathcal{K}, L\in\mathcal{L}}} \left(\sup_{K_\alpha\subseteq K, L_\beta\subseteq L} f_\alpha(\mathbf{x})g_\beta(\mathbf{x}) \right)$$
$$= \max_{\substack{K\cap L=\emptyset\\ K\in\mathcal{K}, L\in\mathcal{L}}} \left(\sup_{K_\alpha\subseteq K} f_\alpha(\mathbf{x}) \sup_{L_\beta\subseteq L} g_\beta(\mathbf{x}) \right) = \max_{\substack{K\cap L=\emptyset\\ K\in\mathcal{K}, L\in\mathcal{L}}} \tilde{f}_K(\mathbf{x})\tilde{g}_L(\mathbf{x}). \tag{15}$$

(c) $\Rightarrow$ (d): Apply $\tilde{f}_{K(\mathbf{x})}(\mathbf{x})\tilde{g}_{L(\mathbf{x})}(\mathbf{x}) \le \max_{K\cap L=\emptyset, K\in\mathcal{K}, L\in\mathcal{L}} \tilde{f}_K(\mathbf{x})\tilde{g}_L(\mathbf{x})$.

$(d) \Rightarrow (a)$: Note that the right hand side of (15) equals $\tilde{f}_{K(\mathbf{x})}(\mathbf{x})\tilde{g}_{L(\mathbf{x})}(\mathbf{x})$ for some $K(\mathbf{x}) \in \mathcal{K}$ and $L(\mathbf{x}) \in \mathcal{L}$ with $K(\mathbf{x}) \cap L(\mathbf{x}) = \emptyset$.
$(b) \Rightarrow (e)$: Apply $\underline{f}_{K(\mathbf{x})}(\mathbf{x})\underline{g}_{L(\mathbf{x})}(\mathbf{x}) \le \max_{K\cap L=\emptyset, K\in\mathcal{K}, L\in\mathcal{L}} \underline{f}_K(\mathbf{x})\underline{g}_L(\mathbf{x})$.
$(e) \Rightarrow (b)$: Use the fact that there exist some disjoint $K(\mathbf{x}) \in \mathcal{K}, L(\mathbf{x}) \in \mathcal{L}$ such that

$$\max_{\substack{K\cap L=\emptyset\\ K\in\mathcal{K}, L\in\mathcal{L}}} \underline{f}_K(\mathbf{x})\underline{g}_L(\mathbf{x}) = \underline{f}_{K(\mathbf{x})}(\mathbf{x})\underline{g}_{L(\mathbf{x})}(\mathbf{x}). \quad \blacksquare \tag{16}$$

Let $\mathcal{F}_C$ be the sigma algebra generated by a collection of sets C. We say $\mathcal{F}_C$ is a finite product sigma sub algebra of $\mathbb{S}$ when

$$\mathcal{C} = \left\{ \prod_{i=1}^{n} A_i, A_i \in \mathcal{C}_i \right\}, \quad \text{with } \mathcal{C}_i \subseteq \mathbb{S}_i \text{ finite for all } i = 1, \ldots, n. \tag{17}$$

It is easy to see that every finite sigma algebra, $\mathcal{F}$, contains a subset G, not containing the empty set, such that every element of $\mathcal{F}$ can be represented uniquely as a disjoint union of elements of G. Call G the disjoint generating set of $\mathcal{F}$.

Our next objective is proving the inequalities of Framework 1, to be accomplished by proving (d) in Lemma 3.6. We start with a simple extension of inequality (3), expressed in terms of indicator functions, from finite spaces to spaces that may not be finite, but which are endowed with a finite product sigma algebra.

Lemma 3.1. *Let Q be any probability product measure on the finite product sigma algebra $\mathcal{F}_{\mathcal{C}}$ with $\mathcal{C}$ given by (17). Then, inequality (a) holds when expectations are taken with respect to Q, and $\{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{B}}$ are $\mathcal{F}_C$ measurable indicator functions.*

Proof: For $i = 1, \ldots, n$, let G_i be the disjoint generating set of $\mathcal{F}_{\mathcal{C}_i}$. By Theorem 1.1, applied on the space $G = \prod_{i=1}^n G_i$,

$$Q(A \square B) \le Q(A)Q(B). \tag{18}$$

Let events A and B be defined by the indicator functions

$$\mathbf{1}_A(\mathbf{x}) = \max_\alpha f_\alpha(\mathbf{x}), \quad \mathbf{1}_B(\mathbf{x}) = \max_\beta g_\beta(\mathbf{x}), \tag{19}$$

and let A_α and B_β be the sets indicated by $f_\alpha(\mathbf{x})$ and $g_\beta(\mathbf{x})$ respectively. Suppose $\mathbf{x} \in S$ satisfies $f_\alpha(\mathbf{x})g_\beta(\mathbf{x}) = 1$ for disjoint α, β. Clearly $A_\alpha \subseteq A$, and as f_α depends on K_α, we have $[\mathbf{x}]_\alpha \subseteq A_\alpha \subseteq A$. As a similar statement holds for B, $\mathbf{x} \in A \square B$, hence,

$$\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{x})g_\beta(\mathbf{x}) \le \mathbf{1}_{A\square B}(\mathbf{x}). \tag{20}$$

Now (20) gives the first inequality below, (18) the second inequality, and (19) the last equality in

$$E_Q\left\{\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\} \le Q(A\square B) \le Q(A)Q(B) = E_Q\left\{\max_\alpha f_\alpha(\mathbf{X})\right\} E_Q\left\{\max_\beta g_\beta(\mathbf{X})\right\}. \blacksquare$$

We say a collection of functions is FP if it generates a finite product sigma algebra contained in $\mathbb{S}$; note that a finite union of FP collections is FP.

Lemma 3.2. *Inequality (a) is true for P any probability product measure on $(S, \mathbb{S})$, and $\{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{B}}$, any finite collections of FP indicator functions.*

Proof: Let $\mathcal{H}$ be the sigma algebra generated by $\{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{B}}$, and $Q := P|_{\mathcal{H}}$, the restriction of P to the finite product sigma algebra $\mathcal{H}$. For h an $\mathcal{H}$ measurable indicator function, that is, for $h(\mathbf{x}) = \mathbf{1}_A(\mathbf{x})$ for some $A \in \mathcal{H}$, we have

$$E_Q h = Q(A) = P(A) = E_P h. \tag{21}$$

Since the product of $\mathcal{H}$ measurable indicators is an $\mathcal{H}$ measurable indicator, and the same is true for the maximum, we have by Lemma 3.1 and (21),

$$\begin{aligned} E_P\left\{\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\} &= E_Q\left\{\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\} \\ &\le E_Q\left\{\max_{\alpha} f_\alpha(\mathbf{X})\right\} E_Q\left\{\max_{\beta} g_\beta(\mathbf{X})\right\} \\ &= E_P\left\{\max_{\alpha} f_\alpha(\mathbf{X})\right\} E_P\left\{\max_{\beta} g_\beta(\mathbf{X})\right\}. \quad \blacksquare \end{aligned}$$

Let $\mathcal{P}$ denote the collection of all product sets of the form $\mathcal{C} = \{\prod_{i=1}^n S_i, S_i \in \mathcal{C}_i\}$ where $\mathcal{C}_i \subseteq \mathbb{S}_i$ are finite for all $i = 1, \ldots, n$. Then

$$\mathbb{S} = \mathcal{F}_{\mathcal{J}} \quad \text{where} \quad \mathcal{J} = \bigcup_{\mathcal{C}\in\mathcal{P}} \mathcal{F}_{\mathcal{C}}. \tag{22}$$

Lemma 3.3 generalizes the inequality from FP indicator functions to $\mathbb{S}$ measurable indicator functions.

Lemma 3.3. *Inequality (a) is true for any probability product measure P and finite collections $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$ of $\mathbb{S}$ measurable indicator functions.*

Proof: For $\mathcal{R}, \mathcal{S}$ subsets of $\mathcal{A}\cup\mathcal{B}$ satisfying $\mathcal{R}\cap\mathcal{S} = \emptyset$ and $\mathcal{R}\cup\mathcal{S} = \mathcal{A}\cup\mathcal{B}$, we proceed by induction on the cardinality of the set $\mathcal{S}$ in the statement $I(\mathcal{R},\mathcal{S})$: inequality (a) is true when $\{f_\alpha\}_{\alpha\in\mathcal{R}\cap\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{R}\cap\mathcal{B}}$ are finite FP collections of indicator functions, and $\{f_\alpha\}_{\alpha\in\mathcal{S}\cap\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{S}\cap\mathcal{B}}$ are any finite collections of $\mathbb{S}$ measurable indicators. Lemma 3.2 shows that $I(\mathcal{A}\cup\mathcal{B}, \emptyset)$ is true, and the conclusion of the present lemma is $I(\emptyset, \mathcal{A}\cup\mathcal{B})$. Assume for some such $\mathcal{R}, \mathcal{S}$ with $\mathcal{S} \ne \mathcal{A}\cup\mathcal{B}$, that $I(\mathcal{R},\mathcal{S})$ is true. For $\gamma \in \mathcal{R}$ with, say $\gamma \in \mathcal{A}$, let $\mathcal{M}$ be the collection of all sets $A \subseteq S$ such that (a) holds for $f_\gamma = \mathbf{1}_A$, and when $\{f_\alpha\}_{\alpha\in\mathcal{R}\cap\mathcal{A}\setminus\{\gamma\}}$ and $\{g_\beta\}_{\beta\in\mathcal{R}\cap\mathcal{B}}$ are finite FP indicators, and $\{f_\alpha\}_{\alpha\in\mathcal{S}\cap\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{S}\cap\mathcal{B}}$ are any collection of $\mathbb{S}$ measurable indicators. The singleton collection f_γ is FP for any $A \in \mathcal{J}$ given by (22). Therefore, for any $A \in \mathcal{J}$, the union $f_\gamma, \{f_\alpha\}_{\alpha\in\mathcal{R}\cap\mathcal{A}\setminus\{\gamma\}}, \{g_\beta\}_{\beta\in\mathcal{R}\cap\mathcal{B}}$ is FP. By the induction hypothesis, $\mathcal{J} \subseteq \mathcal{M}$. Since $\mathcal{M}$ is a monotone class and $\mathcal{J}$ is an algebra which generates $\mathbb{S}$, the monotone class theorem implies $\mathbb{S} \subseteq \mathcal{M}$. This completes the induction. $\blacksquare$

We now relax the requirement that the functions be indicators.

Lemma 3.4. *Inequality (d) is true for any product measure P and finite collections of $\mathbb{S}$ measurable functions $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$ which assume finitely many non-negative values.*

Proof: We prove (d) by induction on m and l, the number of values taken on by the collections $\{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{B}}$, respectively. By Lemma 3.3 inequality (a) is true for finite collections of measurable indicators, and hence by Proposition 1.1, so is inequality (d). Now the base case $m = 2$, $l = 2$ follows readily by extending from indicators to two valued functions by linear transformation.

Assume the result is true for some m and l at least 2, and consider a collection $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ assuming the values $0 \le a_1 < \cdots < a_{m+1}$; a similar argument applies to induct on l. For some k, $2 \le k \le m$, define

$$A_{\alpha,k} = \{\mathbf{x} : f_\alpha(\mathbf{x}) = a_k\},$$

and for $a_{k-1} \le a \le a_{k+1}$, let

$$h_\alpha^a(\mathbf{x}) = f_\alpha(\mathbf{x}) + (a - a_k)\mathbf{1}_{A_{\alpha,k}}(\mathbf{x}),$$

the function f_α with the value of a_k replaced by a. We shall prove that for all $a \in [a_{k-1}, a_{k+1}]$ inequality (d) holds with $\{h_\alpha^a\}_{\alpha\in\mathcal{A}}$ replacing $\{f_\alpha\}_{\alpha\in\mathcal{A}}$. By the induction hypothesis we know it holds at the endpoints, that is, for $a \in \{a_{k-1}, a_{k+1}\}$, since then the collection $\{h_\alpha^a\}_{\alpha\in\mathcal{A}}$ takes on m values; clearly, the case $a = a_k$ suffices to prove the lemma.

Given $\Gamma(\mathbf{x})$, a function with values in $2^{\mathcal{A}}$, with some abuse of notation denote

$$\tilde{f}_{\Gamma(\mathbf{x})}(\mathbf{x}) = \sup_{\alpha:\alpha\in\Gamma(\mathbf{x})} f_\alpha(\mathbf{x}). \tag{23}$$

Note that $\tilde{f}_{K(\mathbf{x})}(\mathbf{x})$ in (9) corresponds to $\Gamma(\mathbf{x}) = \{\alpha : K_\alpha \subseteq K(\mathbf{x})\}$, and similarly for $\tilde{g}_{L(\mathbf{x})}(\mathbf{x})$; for measurability issues see Section 5. For any function $\Gamma(\mathbf{x})$ with values in $2^{\mathcal{A}}$, we have for all $a \in [a_{k-1}, a_{k+1}]$,

$$C_\Gamma := \{\mathbf{x} : \widetilde{h^a}_{\Gamma(\mathbf{x})}(\mathbf{x}) = a,\ \tilde{f}_{\Gamma(\mathbf{x})}(\mathbf{x}) \notin \{a_{k-1}, a_{k+1}\}\} = \{\mathbf{x} : \tilde{f}_{\Gamma(\mathbf{x})}(\mathbf{x}) = a_k\},$$

showing that C_Γ does not depend on a.

Let $D = C_\Gamma$ for $\Gamma(\mathbf{x}) = \mathcal{A}$, and note that

$$\sup_\alpha h_\alpha^a(\mathbf{x}) = a1_D + \sup_\alpha f_\alpha(\mathbf{x})1_{D^c}.$$

Then the right hand side of (d), with $\{h_\alpha^a\}_{\alpha\in\mathcal{A}}$ replacing $\{f_\alpha\}_{\alpha\in\mathcal{A}}$, equals $a\delta + \lambda$, where

$$\delta = P(D)\int \sup_\beta g_\beta(\mathbf{x})dP(\mathbf{x}) \quad \text{and} \quad \lambda = \int_{D^c} \sup_\alpha f_\alpha(\mathbf{x})dP(\mathbf{x}) \int \sup_\beta g_\beta(\mathbf{x})dP(\mathbf{x})$$

do not depend on a. Now, let $E = C_\Gamma$ for $\Gamma(\mathbf{x}) = \{\alpha : K_\alpha = K(\mathbf{x})\}$ and note that $\widetilde{h^a}_{K(\mathbf{x})}(\mathbf{x}) = a1_E + \widetilde{f}_{K(\mathbf{x})}(\mathbf{x})1_{E^c}$. Similarly, the left hand side of (d), with $\{h_\alpha^a\}_{\alpha\in\mathcal{A}}$ replacing $\{f_\alpha\}_{\alpha\in\mathcal{A}}$, equals $a\theta + \eta$, where

$$\theta = \int_E \tilde{g}_{L(\mathbf{x})}(\mathbf{x})dP(\mathbf{x}) \quad \text{and} \quad \eta = \int_{E^c} \widetilde{f}_{K(\mathbf{x})}(\mathbf{x})\tilde{g}_{L(\mathbf{x})}(\mathbf{x})dP(\mathbf{x})$$

do not depend on a. When $a \in \{a_{k-1}, a_{k+1}\}$ the collection $h_\alpha, \alpha \in \mathcal{A}$ takes on m values, so by the induction hypotheses (d) holds with $\{h_\alpha^a\}_{\alpha\in\mathcal{A}}$ replacing $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and we obtain

$$a\theta + \eta \le a\delta + \lambda, \quad \text{for } a \in \{a_{k-1}, a_{k+1}\}. \tag{24}$$

By taking a convex combination, we see that inequality (24) holds for all $a \in [a_{k-1}, a_{k+1}]$, so in particular for a_k, completing the induction. ∎

Lemma 3.5. *Inequality (d) is true for any probability product measure P and finite collections of non-negative $\mathbb{S}$ measurable functions $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$.*

Proof: Lemma 3.4 shows that the result is true for simple functions. By approximating the functions f_α, g_β below by simple functions, $f_{\alpha,k} \uparrow f_\alpha, g_{\beta,k} \uparrow g_\beta$ as $k \uparrow \infty$, and applying the monotone convergence theorem, we have the result for arbitrary non-negative functions. ■

Lemma 3.6. *Inequality (d) is true for countable collections of non-negative $\mathbb{S}$ measurable functions $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$.*

Proof: For $K, L \in 2^{\mathbf{n}}$, let

$$\varphi_K(\mathbf{x}) = \tilde{f}_K(\mathbf{x}) \quad \text{and} \quad \phi_L(\mathbf{x}) = \tilde{g}_K(\mathbf{x}),$$

recalling definition (8). Noting

$$\tilde{f}_{K(\mathbf{x})}(\mathbf{x}) = \sup_{\alpha:K_\alpha\subseteq K(\mathbf{x})} f_\alpha(\mathbf{x}) = \sup_{K\subseteq K(\mathbf{x})} \sup_{\alpha:K_\alpha\subseteq K} f_\alpha(\mathbf{x}) = \sup_{K\subseteq K(\mathbf{x})} \tilde{f}_K(\mathbf{x}) = \tilde{\varphi}_{K(\mathbf{x})}(\mathbf{x}),$$

and

$$\sup_K \varphi_K(\mathbf{x}) = \sup_K \tilde{f}_K(\mathbf{x}) = \sup_\alpha f_\alpha(\mathbf{x}),$$

and similarly for $\{g_\beta\}_{\beta\in\mathcal{B}}$, the result follows immediately upon applying Lemma 3.5 to the finite collections $\{\varphi_K\}_{K\in 2^{\mathbf{n}}}$ and $\{\phi_L\}_{L\in 2^{\mathbf{n}}}$. ■

By Proposition 3.1, at this point we have completed proving all inequalities pertaining to Framework 1. The next proposition connects the two frameworks and completes the proof of Theorem 1.2, and again applying Proposition 3.1, that of Proposition 1.1.

Proposition 3.2. *Inequality (a) holds in Framework 1 for all collections $\{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{B}}$ of given functions, if and only if inequality (b) holds in Framework 2 for all given functions f and g and collections $\mathcal{K}$ and $\mathcal{L}$.*

Proof: $(a) \Rightarrow (b)$. For $L \subseteq \mathbf{n}$ let $P_L(\mathbf{x})$ denote the marginal of P in the coordinates indexed by L. Let functions f, g and collections $\mathcal{K}$ and $\mathcal{L}$ of subsets of $2^{\mathbf{n}}$ be given.

By Fubini's theorem, for any $K \subseteq 2^{\mathbf{n}}$,

$$\begin{aligned} P(\underline{f}_K(\mathbf{x}) \le f(\mathbf{x})) &= \int \mathbf{1}(\underline{f}_K(\mathbf{x}) \le f(\mathbf{x}))dP(\mathbf{x}) = \int\int \mathbf{1}(\underline{f}_K(\mathbf{x}) \le f(\mathbf{x}))dP_{K^c}(\mathbf{x})dP_K(\mathbf{x}) \\ &= \int P_{K^c}(\underline{f}_K(\mathbf{x}) \le f(\mathbf{x}))dP_K(\mathbf{x}) = \int 1 dP_K(\mathbf{x}) = 1, \end{aligned}$$

where the fourth equality holds by definition of the essential infimum. As $\mathcal{K}$ is finite,

$$P(\max_{K\in\mathcal{K}} \underline{f}_K(\mathbf{X}) \le f(\mathbf{X})) = 1, \quad \text{implying} \quad E\left\{\max_{K\in\mathcal{K}} \underline{f}_K(\mathbf{X}\right\} \le E\left\{f(\mathbf{X})\right\},$$

with a similar inequality holding for g. Now we see that (b) holds by applying (a) to the collections $\{\underline{f}_K(\mathbf{x})\}_{K\in\mathcal{K}}$ and $\{\underline{g}_L(\mathbf{x})\}_{L\in\mathcal{L}}$ as in (4).

$(b) \Rightarrow (a)$: Given collections of functions f_α, g_β depending on K_α, L_β, define

$$f(\mathbf{x}) = \sup_\alpha f_\alpha(\mathbf{x}) \quad \text{and} \quad g(\mathbf{x}) = \sup_\beta g_\beta(\mathbf{x}). \tag{25}$$

Now letting $\underline{f}_K, \underline{g}_L$ be as in (4), we have

$$f_\alpha(\mathbf{x}) = \underline{f_\alpha}_{K_\alpha}(\mathbf{x}) \le \underline{f}_{K_\alpha}(\mathbf{x}) \quad \text{and likewise} \quad g_\beta(\mathbf{x}) \le \underline{g}_{L_\beta}(\mathbf{x}).$$

Now, for α, β disjoint,

$$f_\alpha(\mathbf{x}) g_\beta(\mathbf{x}) \le \underline{f}_{K_\alpha}(\mathbf{x}) \underline{g}_{L_\beta}(\mathbf{x}) \le \max_{\substack{K \cap L = \emptyset \\ K \in 2^{\mathbf{n}}, L \in 2^{\mathbf{n}}}} \underline{f}_K(\mathbf{x}) \underline{g}_L(\mathbf{x}).$$

Taking supremum on the left hand side over all disjoint α, β and then expectation, the result now follows by applying inequality (b) and (25). ■

3.2 The Dual Inequality

As observed in [6], the techniques in [10] extend the dual inequality (6) from uniform measure on $\{0,1\}^n \times \{0,1\}^n$ to any product measure on a discrete finite product space S. Specifically, Lemmas 3.2(iii), 3.4, and 3.5 of [10] carry over with minimal changes, essentially by replacing $\Box$ by $\Diamond$ and $\cap$ by $\times$ appropriately; for example, the dual version of Lemma 3.4 would begin with the identity

$$(f \times f)^{-1}(A \Diamond B) = \bigcup_{C_1, C_2} \{(f \times f)^{-1}(C_1 \times C_2)\}$$

where the union is over all C_1, C_2 such that C_1 is a maximal cylinder of A, C_2 is a maximal cylinder of B, and $C_1 \perp C_2$; see Sections 3 and 2 of [10] for the formal definitions of maximal cylinder, and perpendicularity $\perp$, respectively.

Now the proof of Theorem 1.5 and Proposition 1.2 follow in a nearly identical manner to that of Theorem 1.2 and Proposition 1.1. For instance, to prove (d$'$), consider

$$C_\Gamma = \{(\mathbf{x}, \mathbf{y}) : \widetilde{h^a}_{\Gamma(\mathbf{x},\mathbf{y})}(\mathbf{x}) = a,\ \tilde{f}_{\Gamma(\mathbf{x},\mathbf{y})}(\mathbf{x}) \notin \{a_{k-1}, a_{k+1}\}\} = \{(\mathbf{x}, \mathbf{y}) : \tilde{f}_{\Gamma(\mathbf{x},\mathbf{y})}(\mathbf{x}) = a_k\}.$$

Setting $D = C_\Gamma$ for $\Gamma(\mathbf{x}, \mathbf{y}) = \{\alpha : K_\alpha = K(\mathbf{x}, \mathbf{y})\}$ we can write the left hand side of (d$'$) as $a\theta + \eta$, with

$$\theta = \int_D \tilde{g}_{L(\mathbf{x},\mathbf{y})}(\mathbf{y}) dP(\mathbf{x}) dP(\mathbf{y}) \quad \text{and} \quad \eta = \int_{D^c} \widetilde{f}_{K(\mathbf{x},\mathbf{y})}(\mathbf{x}) \tilde{g}_{L(\mathbf{x},\mathbf{y})}(\mathbf{y}) dP(\mathbf{x}) dP(\mathbf{y}),$$

and using $E = C_\Gamma$ for $\Gamma = \mathcal{A}$, the right hand side becomes $a\delta + \lambda$, where

$$\delta = \int_D \sup_\beta g_\beta(\mathbf{x}) dP(\mathbf{x}) \quad \text{and} \quad \lambda = \int_{D^c} \sup_{\alpha,\beta} f_\alpha(\mathbf{x}) g_\beta(\mathbf{x}) dP(\mathbf{x})$$

with θ, η, δ and λ not depending on a.

4 A PQD ordering inequality

Consider a collection $\{f_\alpha(\mathbf{x})\}_{\alpha=1}^m$ of functions which are all increasing or all decreasing in each component of $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Let $\mathbf{X} = (X_1, \ldots, X_n) \in \mathbf{R}^n$ be a vector of independent random variables, $\mathbf{Y} = (Y_1, \ldots, Y_n)$ an independent copy of $\mathbf{X}$, and for each $\alpha = 1, \ldots, m$, let $H_\alpha \subseteq \mathbf{n}$, and

$$\mathbf{Z}_\alpha = (Z_{1,\alpha}, \ldots, Z_{n,\alpha}), \tag{26}$$

where $Z_{i,\alpha} = Y_i$ if $i \in H_\alpha$, and $Z_{i,\alpha} = X_i$, if $i \notin H_\alpha$. Now let

$$\mathbf{U} = (f_1(\mathbf{Z}_1), \ldots, f_m(\mathbf{Z}_m)) \quad \text{and} \quad \mathbf{V} = (f_1(\mathbf{X}), \ldots, f_m(\mathbf{X})). \tag{27}$$

Inequalities between vectors below are coordinate-wise. When (28) below holds, we say that the components of $\mathbf{V}$ are more 'Positively Quadrant Dependent' than those of $\mathbf{U}$, and write $\mathbf{U} \le_{PQD} \mathbf{V}$.

Theorem 4.1. *For every* $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbf{R}^m$ *and* $H_\alpha \subseteq \mathbf{n}, \alpha = 1, \ldots, m$,

$$P(\mathbf{U} \ge \mathbf{c}) \le P(\mathbf{V} \ge \mathbf{c}) \quad \textit{and} \quad P(\mathbf{U} \le \mathbf{c}) \le P(\mathbf{V} \le \mathbf{c}). \tag{28}$$

Proof: Since (28) holds for $\mathbf{U}, \mathbf{V}$ if and only if it holds for $-\mathbf{U}, -\mathbf{V}$, by replacing the collection $\{f_\alpha(\mathbf{x})\}_{\alpha=1}^m$ by $\{-f_\alpha(\mathbf{x})\}_{\alpha=1}^m$ when the functions are decreasing, it suffices to consider the increasing case.

For $k \in \{0, \ldots, n\}$ let $H_\alpha^k = H_\alpha \cap \{0, \ldots, k\}$, and with H_α replaced by H_α^k, let $\mathbf{Z}_\alpha^k$ and $\mathbf{U}^k$ be defined as in (26) and (27) respectively. We prove the first inequality in (28) by induction on k in

$$P(\mathbf{U}^k \ge \mathbf{c}) \le P(\mathbf{V} \ge \mathbf{c}); \tag{29}$$

the second inequality in (28) follows in the same manner. Inequality (29) is trivially true, with equality, when $k = 0$, since then $H_\alpha^k = \emptyset$ and $\mathbf{Z}_\alpha = \mathbf{X}$ for all $\alpha \in \mathbf{m}$. Now assume inequality (29) is true for $0 \le k < n$ and set

$$B = \{\alpha : k + 1 \in H_\alpha\}.$$

Then

$$\begin{aligned}
& P(\mathbf{U}^{k+1} \ge \mathbf{c}) \\
= \; & P(f_1(\mathbf{Z}_1^{k+1}) \ge c_1, \ldots, f_m(\mathbf{Z}_m^{k+1}) \ge c_m) \\
= \; & E[P(f_1(\mathbf{Z}_1^{k+1}) \ge c_1, \ldots, f_m(\mathbf{Z}_m^{k+1}) \ge c_m | X_l, Y_l, l \ne k+1)] \\
= \; & E[P(f_\alpha(\mathbf{Z}_\alpha^k) \ge c_\alpha, \alpha \notin B | X_l, Y_l, l \ne k+1) P(f_\alpha(\mathbf{Z}_\alpha^{k+1}) \ge c_\alpha, \alpha \in B | X_l, Y_l, l \ne k+1)] \\
= \; & E[P(f_\alpha(\mathbf{Z}_\alpha^k) \ge c_\alpha, \alpha \notin B | X_l, Y_l, l \ne k+1) P(f_\alpha(\mathbf{Z}_\alpha^k) \ge c_\alpha, \alpha \in B | X_l, Y_l, l \ne k+1)] \\
\le \; & E[P(f_1(\mathbf{Z}_1^k) \ge c_1, \ldots, f_m(\mathbf{Z}_m^k) \ge c_m) | X_l, Y_l, l \ne k+1] \\
= \; & P(\mathbf{U}^k \ge \mathbf{c}) \\
\le \; & P(\mathbf{V} \ge \mathbf{c}),
\end{aligned}$$

where the third equality follows from the independence of X_{k+1} and Y_{k+1} and the fourth from the fact that $\{f_\alpha(\mathbf{Z}_\alpha^k)\}_{\alpha \in B}$ has the same conditional distribution when either X_{k+1} or Y_{k+1}

appears as the $k+1^{st}$ coordinate of the $\mathbf{Z}$ vector; the first inequality follows from the fact that conditioned on $X_l, Y_l, l \neq k+1$, the functions $f_\alpha(\mathbf{Z}_\alpha^k)$ are all increasing in X_{k+1} and are therefore (conditionally) associated, and the second inequality is the induction hypothesis (29). In fact, for the first inequality above it suffices to see that the product of the two probabilities conditioned on $X_l, Y_l, l \neq k+1$ is the product of (conditional) expectations of two increasing functions of X_{k+1}, which is smaller than the (conditional) expectation of the product. ■

Taking $\mathbf{c} = (c, \ldots, c)$ we immediately have

Corollary 4.1. *For all* $c \in \mathbf{R}$,

$$P(\max_\alpha f_\alpha(\mathbf{Z}_\alpha) \leq c) \leq P(\max_\alpha f_\alpha(\mathbf{X}) \leq c) \quad \text{or equivalently} \quad \max_\alpha f_\alpha(\mathbf{X}) \leq_{ST} \max_\alpha f_\alpha(\mathbf{Z}_\alpha).$$

Application 1. Consider the framework of Theorem 1.2, with $f_\alpha(\mathbf{x}), g_\beta(\mathbf{x}), \alpha \in \mathcal{A}, \beta \in \mathcal{B}$ all increasing or all decreasing functions which depend on coordinates K_α, L_β. Define $\mathcal{D}$ to be a collection of functions

$$\mathcal{D} = \{f_\alpha(\mathbf{X}) + g_\beta(\mathbf{X}) : K_\alpha \cap L_\beta = \emptyset\},$$

and for $\mathbf{Y} = (Y_1, \ldots, Y_n)$ as above, set

$$\mathcal{D}^* = \{f_\alpha(\mathbf{X}) + g_\beta(\mathbf{Y}) : K_\alpha \cap L_\beta = \emptyset\}.$$

By Theorem 4.1 we have

$$\mathcal{D}^* \leq_{PQD} \mathcal{D}.$$

Applying Corollary 4.1,

$$\max_{\alpha \cap \beta = \emptyset} \{f_\alpha(\mathbf{X}) + g_\beta(\mathbf{X})\} \leq_{ST} \max_{\alpha \cap \beta = \emptyset} \{f_\alpha(\mathbf{X}) + g_\beta(\mathbf{Y})\}.$$

Exponentiating the last relation and replacing e^{f_α} by f_α, using obvious properties of the max, we obtain

$$\max_{\alpha \cap \beta = \emptyset} \{f_\alpha(\mathbf{X}) g_\beta(\mathbf{X})\} \leq_{ST} \max_{\alpha \cap \beta = \emptyset} \{f_\alpha(\mathbf{X}) g_\beta(\mathbf{Y})\}. \tag{30}$$

and therefore

$$E\{\max_{\alpha \cap \beta = \emptyset} f_\alpha(\mathbf{X}) g_\beta(\mathbf{X})\} \leq E\{\max_{\alpha \cap \beta = \emptyset} f_\alpha(\mathbf{X}) g_\beta(\mathbf{Y})\} \leq E\{\max_\alpha f(\mathbf{X})\}\, E\{\max_\beta g(\mathbf{X})\},$$

for nonnegative monotone functions f_α and g_β. Thus the relation (30) is stronger than the BKR inequality for monotone sets, which was proved in [11]. Alexander [1] presents similar functional versions in this context.

As an example we return to order statistics as in Section 2.1. From (30) we derive, for example, that

$$X_{[1]} X_{[2]} \leq_{ST} X_{[1]} Y_{[2]} \vee Y_{[1]} X_{[2]}.$$

Generalizing by using the functions (11), we obtain for any $p + q = m$,

$$\prod_{j=1}^{m} X_{[j]} \leq_{ST} \max_{\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \{1, \ldots, m\}} \prod X_{[i_q]} Y_{[j_q]}.$$

5 Appendix on Measurability

In this section we briefly deal with various measurability issues. The measurability of the functions defined in (9) can be seen from

$$\tilde{f}_{K(\mathbf{x})}(\mathbf{x}) = \sum_K \tilde{f}_K(\mathbf{x})\mathbf{1}(K(\mathbf{x}) = K),$$

since the given function $K(\mathbf{x})$ is assumed measurable. Similarly for (23),

$$\tilde{f}_{\Gamma(\mathbf{x})}(\mathbf{x}) = \sum_{A \in 2^{\mathcal{A}}} \sup_{\alpha \in A} f_\alpha(\mathbf{x})\mathbf{1}(\Gamma(\mathbf{x}) = A).$$

We next prove that given a non-negative, $(\mathbb{S}, \mathbb{B})$ measurable function $f : S \to \mathbf{R}$ and any $K \subseteq \mathbf{n}$, the function $\underline{f}_K(\mathbf{x})$ defined in (4) is $(\mathbb{S}, \mathbb{B})$ measurable. Letting

$$f_r(\mathbf{x}) = \min(f(\mathbf{x}), r)$$

and $P_L(\mathbf{x})$ be the marginal of $P(\mathbf{x})$ on the coordinates $\mathbf{x}_L$, we have

$$\lim_{p\to\infty} \left(\int (r - f_r(\mathbf{x}))^p dP_{K^c}(\mathbf{x}) \right)^{1/p} = \operatorname{ess} \sup_{\mathbf{y} \in [\mathbf{x}]_K} (r - f_r(\mathbf{y})) = r - \operatorname{ess} \inf_{\mathbf{y} \in [\mathbf{x}]_K} f_r(\mathbf{y}).$$

Tonelli's theorem (see e.g. [4]) now implies that $\operatorname{ess} \inf_{\mathbf{y} \in [\mathbf{x}]_K} f_r(\mathbf{y})$ is measurable. Letting $r \uparrow \infty$ shows that (4) is measurable.

The only complication regarding measurability of the pair $(K(\mathbf{x}), L(\mathbf{x}))$ in (16) is that the maximum may not be uniquely attained, since otherwise we would simply have

$$\{\mathbf{x} : K(\mathbf{x}) = K, L(\mathbf{x}) = L\} = \bigcap_{K' \cap L' = \emptyset} \{\mathbf{x} : \underline{f}_K(\mathbf{x})\underline{g}_L(\mathbf{x}) \geq \underline{f}_{K'}(\mathbf{x})\underline{g}_{L'}(\mathbf{x})\},$$

a finite intersection of measurable sets, so measurable. To handle the problem of non-uniqueness, let $\prec$ be an arbitrary total order on the finite collection of subsets of $\mathbf{n} \times \mathbf{n}$, so that when the max is not unique we can choose $(K(\mathbf{x}), L(\mathbf{x}))$ to be the first disjoint pair that attains the maximum. Then $\{\mathbf{x} : K(\mathbf{x}) = K, L(\mathbf{x}) = L\} = F \cap G$ where

$$F = \bigcap_{\substack{(K',L') \prec (K,L) \\ K' \cap L' = \emptyset}} \{\mathbf{x} : \underline{f}_K(\mathbf{x})\underline{g}_L(\mathbf{x}) > \underline{f}_{K'}(\mathbf{x})\underline{g}_{L'}(\mathbf{x})\}$$

and

$$G = \bigcap_{\substack{(K',L') \succeq (K,L) \\ K' \cap L' = \emptyset}} \{\mathbf{x} : \underline{f}_K(\mathbf{x})\underline{g}_L(\mathbf{x}) \geq \underline{f}_{K'}(\mathbf{x})\underline{g}_{L'}(\mathbf{x})\}$$

and again measurability follows. Similar remarks apply to the maximizing $K(\mathbf{x}), L(\mathbf{x})$ in (15) for the implication $(d) \Rightarrow (a)$.

Finally, we note inequality (a), and therefore also (b), holds on the completion of $(S, \overline{\mathbb{S}})$ of $(S, \mathbb{S})$ with respect to P. Proposition 2.12 of [4] shows that for every $\overline{\mathbb{S}}$ measuable function $\overline{f}$ there exists an $\mathbb{S}$ measurable function f such that $\overline{f} = f$ with (completed) measure one. Hence, replacing all $\overline{\mathbb{S}}$ measurable functions in (a) by their $\mathbb{S}$ measurable counterparts and applying (a) over the space $(S, \mathbb{S})$ shows (a) holds on $(S, \overline{\mathbb{S}})$.

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